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Group solution for an unsteady non-Newtonian Hiemenz flow with variable fluid properties and suction/injection

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The theoretic transformation group approach is applied to address the problem of unsteady boundary layer flow of a non-Newtonian fluid near a stagnation point with variable viscosity and thermal conductivity. The application of a two-parameter group method reduces the number of independent variables by two, and consequently the governing partial differential equations with the boundary conditions transformed into a system of ordinary differential equations with the appropriate corresponding conditions. Two systems of ordinary differential equations have been solved numerically using a fourth-order Runge–Kutta algorithm with a shooting technique. The effects of various parameters governing the problem are investigated.

Keywords: non-Newtonian fluid, stagnation point, two-parameter group method, variable viscosity

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1. Introduction

Recently, non-Newtonian fluids have attracted considerable attention in many activities due to their theoretical and technical applications in modern technology, such as petroleum reservoirs, ground-water hydrology, nuclear waste disposal, geothermal energy production, biological fluids transpiration cooling, design of solid, catalytic reactors, and food industry. Schowalter[1] was the one who first studied the boundary-layer flow of a non-Newtonian fluid. Several books excellently summarized the field of non-Newtonian fluids, including Astarita and Marrucci,[2] Schowalter,[3] Crochet *et al.*,[4] and Bird *et al.*[5] These fluids have a common character that the stress–strain relationship is nonlinear.

Several types of non-Newtonian fluids exist, and the most common type is the power law fluid, the shear stress of which is given by $\tau = \mu(\partial u/\partial y)^n$. The problem of laminar flows of power-law non-Newtonian fluids has been studied by several authors. Gupta *et al.*[6] analyzed the problem of steady flow of a non-Newtonian fluid past an infinite porous flat plate subject to suction. Zhang and Wang[7] transformed the magnetohydrodynamic boundary layer system for an electrically conducting power-law fluid with certain boundary conditions into a boundary value problem of a third-order nonlinear ordinary differential equation. They established the uniqueness, existence, and nonexistence of self-similar solutions by a rigorous mathematical analysis method. Olajuwon[8] examined the steady temperature field of a reacting non-Newtonian power-law fluid caused by the exothermic reaction of the fluid molecules as it flows in the presence of the thermal radiation over a flat plate.

The stagnation flows have many applications, such as flows over the tips of rockets, aircrafts, submarines, and oil ships. In addition, the introduction of time as the third independent variable in the unsteady problem increases the complexity of the problem. Much attempt has been made to find analytical and numerical solutions by applying certain special conditions and different mathematical approaches. The problem of unsteady stagnation point flow has been extended in various ways. Soundalgekar *et al.*[9] studied the effect of variable wall temperature on the flow of MHD heat transfer with unsteady stagnation point. Labropulu[10] investigated the unsteady two-dimensional stagnation-point flow of a viscous fluid impinging on an infinite plate in the presence of a transverse magnetic field. Fang *et al.*[11] solved the problem of unsteady boundary layer of incompressible stagnation-point flow with mass transfer using a similarity transformation technique. Xu *et al.*[12] applied the Homotopy analysis method to study the unsteady boundary layer flow of a power-law non-Newtonian fluid near the forward stagnation point.

In all the above-mentioned studies, the physical properties of the fluid were assumed to be constant. Actually, they change significantly with temperature. Many authors have studied the effect of variable properties in the studied fluids.[13–17]

Motivated by all the previous studies, we investigate the effects of variable viscosity and thermal conductivity on an unsteady power-law laminar flow near a time-dependent stagnation point with suction/blowing.

In the present analysis, we perform the two-parameter group transformation to reduce the system of partial differ-

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ential equations to a system of ordinary differential equations. The two-parameter group method can reduce the number of independent variables by two, and consequently the system of governing partial differential equations with the boundary and initial conditions reduces to a system of ordinary differential equations with appropriate corresponding conditions.[18] The governing boundary layer equations have been transformed to ordinary differential equations via group analysis. The transformed ordinary differential equations have been solved numerically by using the fourth-order Runge–Kutta scheme with the shooting technique.

2. Flow analysis

Consider an unsteady, incompressible, laminar flow of a non-Newtonian power-law fluid with variable properties near a stagnation point in the boundary layer between a free stream and a horizontal flat plate. The x direction is along the plate, the y direction is normal to it, and u, v denote the respective velocity. It is assumed that when $t < 0$, the surface of the body and the fluid are at rest, then at time $t = 0$, the external stream is set into an impulsive motion from rest with the velocity $U(x,t)$. The surface temperature distribution is a function of the distance x and time t, i.e., $T_{\rm w}(x,t)$, and far from the plate, the fluid is isothermal of constant temperature T_∞ (see Fig. 1). Under these conditions and the boundary layer approximation the governing equations describing the problem are [19]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y}\right)^n\right), \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\rho C_{\rm p}}\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right), \tag{3}$$

with boundary conditions

$$t<0: u=v=0,\ \ T=T_\infty,\ \ \text{everywhere}, \tag{4a}$$

$$t\geq 0: u=0,\ v=v_{\rm w}=\text{const.},\ T=T_{\rm w}(x,t),\ \text{at } y=0, \tag{4b}$$

$$u=U(x,t),\ \ T=T_\infty,\ \ \text{as}\ \ y\to\infty, \tag{4c}$$

where $v_{\rm w}$ is the constant velocity normal to the plate, ρ is the density of the fluid, and $C_{\rm p}$ the specific heat at constant pressure. Using the following non-dimensional variables:

$$\tilde{x}=\frac{x}{L},\ \ \tilde{y}=\frac{y}{L}\mathrm{Re}^{1/(n+1)},\ \ \tilde{t}=\frac{tU^*}{L},\ \ \tilde{u}=\frac{u}{U^*},$$

$$\tilde{v}=\frac{v}{U^*}\mathrm{Re}^{1/(n+1)},\ \ \tilde{U}=\frac{U}{U^*},\ \ \tilde{\mu}=\frac{\mu}{\mu_\infty},\ \ \tilde{\kappa}=\frac{\kappa}{\kappa_\infty},$$

$$\theta=\frac{T-T_\infty}{T_1},\ \ T_1=T_{\rm w}(x,t)-T_\infty, \tag{5}$$

where L, U^*, μ_∞, and κ_∞ are the characteristic length, velocity, viscosity, and thermal conductivity, respectively, $\mathrm{Re} = L^n\rho(U^*)^{2-n}/\mu_\infty$ is the Reynolds number. We also define the stream function ψ as

$$\tilde{u}=\frac{\partial\psi}{\partial\tilde{y}},\ \ \tilde{v}=-\frac{\partial\psi}{\partial\tilde{x}},$$

then equations (1)–(4) will have the form

$$\frac{\partial^2\psi}{\partial y\partial t}+\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial\psi}{\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} =\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\mu n\left(\frac{\partial^2\psi}{\partial y^2}\right)^{n-1}\frac{\partial^3\psi}{\partial y^3}+\left(\frac{\partial^2\psi}{\partial y^2}\right)^n\frac{\partial\mu}{\partial y}, \tag{6}$$

$$\theta\frac{\partial T_1}{\partial t}+T_1\frac{\partial\theta}{\partial t}+\frac{\partial\psi}{\partial y}\left(\theta\frac{\partial T_1}{\partial x}+T_1\frac{\partial\theta}{\partial x}\right)-T_1\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} =\frac{T_1}{\mathrm{Pr}}\left(\kappa\frac{\partial^2\theta}{\partial y^2}+\frac{\partial\kappa}{\partial y}\frac{\partial\theta}{\partial y}\right), \tag{7}$$

where $\mathrm{Pr}=\rho C_{\rm p}U^*L/\kappa_\infty\mathrm{Re}^{2/(n+1)}$ is the Prandtl number, the corresponding boundary conditions become

$$t<0: \frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial y}=0,\ \ \theta=0,\ \ \text{everywhere}, \tag{8a}$$

$$t\geq 0: \frac{\partial\psi}{\partial y}=0,\ \ \frac{\partial\psi}{\partial x}=-v_{\rm w},\ \ \theta=1,\ \ \text{at}\ \ y=0, \tag{8b}$$

$$\frac{\partial\psi}{\partial y}=U(x,t),\ \ \theta=0,\ \ \text{as}\ \ y\to\infty. \tag{8c}$$

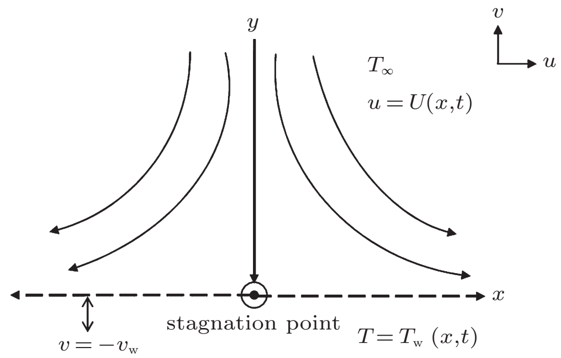

Fig. 1. Physical model and system.

3. Group analysis and similarity equations

The method of solution depends on the application of two parametric group of transformation to the system of partial differential equations (6)–(8). Under this transformation the three independent variables (x,y,t) will be reduced by two and differential equations (6)–(8) transform into ordinary differential equations.

3.1. The group systematic formulation

The procedure is initiated with the group G, a class of transformation of two parameters (a_1,a_2) of the form

$$G: \bar{S}=C^S(a_1,a_2)S+K^S(a_1,a_2), \tag{9}$$

where S stands for $x, y, t, \psi, T_1, \theta, U, \mu$, and κ. The real-valued functions C^s and K^s are at least differentiable in their real arguments (a_1, a_2).

3.2. The invariance analysis

Transformation of the differential equations (6)–(8) required transformations for their partial derivatives. The transformation of derivatives are obtained from G via chain rule operations

$$\bar{S}_{\bar{i}} = \left(\frac{C^S}{C^i}\right) S_i,\ \ \bar{S}_{\bar{i}\bar{j}} = \left(\frac{C^S}{C^i C^j}\right) S_{ij},$$
$$\bar{S}_{\bar{i}\bar{j}\bar{k}} = \left(\frac{C^S}{C^i C^j C^k}\right) S_{ijk}, \tag{10}$$

where $i = x, y, t$, $j = x, y, t$, $k = x, y, t$, and S stands for $\psi, \theta, T_1, U, \mu$, and κ.

Equations (6) and (7) are said to be invariantly transformed under Eqs. (9) and (10) whenever

$$\begin{aligned}
&\frac{\partial^2 \bar{\psi}}{\partial \bar{y} \partial \bar{t}} + \frac{\partial^2 \bar{\psi}}{\partial \bar{x} \partial \bar{y}} \frac{\partial \bar{\psi}}{\partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} - \frac{\partial \bar{U}}{\partial \bar{t}} - \bar{U} \frac{\partial \bar{U}}{\partial \bar{x}} \\
&- \bar{\mu} n \left(\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2}\right)^{n-1} \frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3} - \left(\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2}\right)^n \frac{\partial \bar{\mu}}{\partial \bar{y}} \\
= {} & H_1(a_1, a_2) \left[\frac{\partial^2 \psi}{\partial y \partial t} + \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial U}{\partial t}\right. \\
&\left. - U \frac{\partial U}{\partial x} - \mu n \left(\frac{\partial^2 \psi}{\partial y^2}\right)^{n-1} \frac{\partial^3 \psi}{\partial y^3} - \left(\frac{\partial^2 \psi}{\partial y^2}\right)^n \frac{\partial \mu}{\partial y}\right], \qquad (11) \\
&\bar{\theta} \frac{\partial \bar{T}_1}{\partial \bar{t}} + \bar{T}_1 \frac{\partial \bar{\theta}}{\partial \bar{t}} + \frac{\partial \bar{\psi}}{\partial \bar{y}} (\bar{\theta} \frac{\partial \bar{T}_1}{\partial \bar{x}} + \bar{T}_1 \frac{\partial \bar{\theta}}{\partial \bar{x}}) \\
&- \bar{T}_1 \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial \bar{\theta}}{\partial \bar{y}} - \frac{\bar{T}_1}{\mathrm{Pr}} \left(\bar{\kappa} \frac{\partial^2 \bar{\theta}}{\partial \bar{y}^2} + \frac{\partial \bar{\kappa}}{\partial \bar{y}} \frac{\partial \bar{\theta}}{\partial \bar{y}}\right) \\
= {} & H_2(a_1, a_2) \left[\theta \frac{\partial T_1}{\partial t} + T_1 \frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y} \left(\theta \frac{\partial T_1}{\partial x} + T_1 \frac{\partial \theta}{\partial x}\right)\right. \\
&\left. - T_1 \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} - \frac{T_1}{\mathrm{Pr}} (\kappa \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial \kappa}{\partial y} \frac{\partial \theta}{\partial y})\right], \qquad (12)
\end{aligned}$$

where $H_1(a_1, a_2)$, $H_2(a_1, a_2)$ may be constants. Substituting Eqs. (9) and (10) into Eqs. (6) and (7), we obtain

$$\begin{aligned}
&\frac{\partial^2 \bar{\psi}}{\partial \bar{y} \partial \bar{t}} + \frac{\partial^2 \bar{\psi}}{\partial \bar{x} \partial \bar{y}} \frac{\partial \bar{\psi}}{\partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} - \frac{\partial \bar{U}}{\partial \bar{t}} - \bar{U} \frac{\partial \bar{U}}{\partial \bar{x}} \\
&- \bar{\mu} n \left(\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2}\right)^{n-1} \frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3} - \left(\frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2}\right)^n \frac{\partial \bar{\mu}}{\partial \bar{y}} \\
&- R_1(a_1, a_2) \\
= {} & H_1(a_1, a_2) \left[\frac{\partial^2 \psi}{\partial y \partial t} + \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial U}{\partial t}\right. \\
&\left. - U \frac{\partial U}{\partial x} - \mu n \left(\frac{\partial^2 \psi}{\partial y^2}\right)^{n-1} \frac{\partial^3 \psi}{\partial y^3} - \left(\frac{\partial^2 \psi}{\partial y^2}\right)^n \frac{\partial \mu}{\partial y}\right], \qquad (13) \\
&\bar{\theta} \frac{\partial \bar{T}_1}{\partial \bar{t}} + \bar{T}_1 \frac{\partial \bar{\theta}}{\partial \bar{t}} + \frac{\partial \bar{\psi}}{\partial \bar{y}} \left(\bar{\theta} \frac{\partial \bar{T}_1}{\partial \bar{x}} + \bar{T}_1 \frac{\partial \bar{\theta}}{\partial \bar{x}}\right) - \bar{T}_1 \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial \bar{\theta}}{\partial \bar{y}} \\
&- \frac{\bar{T}_1}{\mathrm{Pr}} \left(\bar{\kappa} \frac{\partial^2 \bar{\theta}}{\partial \bar{y}^2} + \frac{\partial \bar{\kappa}}{\partial \bar{y}} \frac{\partial \bar{\theta}}{\partial \bar{y}}\right) - R_2(a_1, a_2) \\
= {} & H_2(a_1, a_2) \left[\theta \frac{\partial T_1}{\partial t} + T_1 \frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y} \left(\theta \frac{\partial T_1}{\partial x} + T_1 \frac{\partial \theta}{\partial x}\right)\right. \\
&\left. - T_1 \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} - \frac{T_1}{\mathrm{Pr}} \left(\kappa \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial \kappa}{\partial y} \frac{\partial \theta}{\partial y}\right)\right], \qquad (14)
\end{aligned}$$

where

$$\begin{aligned}
R_1(a_1, a_2) &= -\frac{K^\mu (C^\Psi)^n}{(C^y)^{2n+1}} (\frac{\partial^2 \psi}{\partial y^2})^{n-1} \frac{\partial^3 \psi}{\partial y^3} - \frac{K^U C^U}{C^x} \frac{\partial U}{\partial x}, \qquad (15a) \\
R_2(a_1, a_2) &= \frac{K^\theta C^{T_1}}{C^t} \frac{\partial T_1}{\partial t} + \frac{K^{T_1} C^\theta}{C^t} \frac{\partial \theta}{\partial t} + \frac{K^\theta C^\Psi C^{T_1}}{C^x C^y} \frac{\partial \psi}{\partial y} \frac{\partial T_1}{\partial x} \\
&+ \frac{K^{T_1} C^\Psi C^\theta}{C^x C^y} (\frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y}) \\
&- \frac{C^\theta}{Pr (C^y)^2} \frac{\partial^2 \theta}{\partial y^2} (K^{T_1} C^\kappa \kappa + K^\kappa C^{T_1} + K^{T_1} K^\kappa) \\
&- \frac{K^{T_1} C^\kappa C^\theta}{Pr (C^y)^2} \frac{\partial \kappa}{\partial y} \frac{\partial \theta}{\partial y}. \qquad (15b)
\end{aligned}$$

The invariance conditions (11) and (12) imply that $R_1(a_1, a_2) \equiv 0, R_2(a_1, a_2) \equiv 0$, thus

$$\begin{aligned}
H_1(a_1, a_2) &= C^\Psi / C^y C^t = C^\Psi / C^x (C^y)^2 = C^U / C^t \\
&= (C^U)^2 / C^x = C^\mu (C^\Psi)^n / (C^y)^{2n+1}, \qquad (16a) \\
H_2(a_1, a_2) &= C^\theta C^{T_1} / C^t = C^\Psi C^{T_1} C^\theta / C^x C^y \\
&= C^\kappa C^{T_1} C^\theta / (C^y)^2. \qquad (16b)
\end{aligned}$$

From Eqs. (15a) and (15b), the vanishing of R_1 and R_2 implies that

$$K^\mu = K^\kappa = K^\theta = K^U = 0. \tag{17}$$

From Eqs. (16a) and (16b) and invoking the results (17), we obtain

$$\begin{aligned}
C^\mu &= \frac{(C^y)^{n+1} (C^t)^{n-2}}{(C^x)^{n-1}},\ \ C^\Psi = \frac{C^x C^y}{C^t}, \\
C^U &= \frac{C^x}{C^t},\ \ C^\kappa = \frac{(C^y)^2}{C^t}. \qquad (18)
\end{aligned}$$

Moreover, the boundary conditions (8) must be invariant under the same transformation, which implies

$$K^y = 0,\ \ C^\theta = 1. \tag{19}$$

By substituting Eqs. (17)–(19) into Eq. (9), we obtain the class G of two-parameter group under which the system of partial differential equations and boundary conditions (6)–(8) are transformed invariantly, which are given by

$$G:\begin{cases}\bar{S}\begin{cases}\bar{x}=C^x x+K^x,\\ \bar{y}=C^y y,\\ \bar{t}=C^t t+K^t,\end{cases}\\ \bar{\psi}=\dfrac{C^x C^y}{C^t}\psi+K^\psi,\\ \bar{\theta}=\theta,\\ \bar{T}_1=C^{T_1}T_1,\\ \bar{U}=\dfrac{C^x}{C^t}U,\\ \bar{\mu}=\dfrac{(C^y)^{n+1}(C^t)^{n-2}}{(C^x)^{n-1}}\mu,\\ \bar{\kappa}=\dfrac{(C^y)^2}{C^t}\kappa.\end{cases} \tag{20}$$

This group possesses complete sets of absolute invariants $\eta(x,y,t)$ and $g_i(x,y,t,\psi,\theta,T_1,U,\mu,\kappa)$ $(i=1,2,\ldots,6)$, which are the six absolute invariants corresponding to ψ, θ, T_1, U, μ, and κ.

3.3. The complete set of absolute invariants

The complete set of absolute invariants consists of the absolute invariant of the dependant variables (similarity variable) and six absolute invariants corresponding to dependant variables ψ, θ, T_1, μ, U, and κ. If $\eta=\eta(x,y,t)$ is the absolute invariant of the independent variables, then

$$g_i(x,y,t,\psi,\theta,T_1,U,\mu,\kappa)=F_i(\eta(x,y,t)),\quad i=1,2,\ldots,6,$$

are the dependant absolute invariants. The application of a basic theory in group theory states that a function $g_i(x,y,t,\psi,\theta,T_1,U,\mu,\kappa)$ is an absolutely invariant if it satisfies the following two first-order linear differential equations:[20]

$$\sum_{i=1}^{9}(\alpha_i\xi_i+\alpha_{i+1})\frac{\partial g_i}{\partial\xi_i}=0,$$
$$\sum_{i=1}^{9}(\beta_i\xi_i+\beta_{i+1})\frac{\partial g_i}{\partial\xi_i}=0, \tag{21}$$

where ξ_i stands for $x,y,t;\psi,\theta,T_1,U,\mu$, and κ, respectively, and α's and β's are defined by the relations

$$\alpha_1=\left[\frac{\partial C^x}{\partial a_1}\right]_{(a_1^0,a_2^0)},\quad \alpha_2=\left[\frac{\partial K^x}{\partial a_1}\right]_{(a_1^0,a_2^0)},$$
$$\alpha_3=\left[\frac{\partial C^y}{\partial a_1}\right]_{(a_1^0,a_2^0)},\quad \ldots\quad \beta_1=\left[\frac{\partial C^x}{\partial a_2}\right]_{(a_1^0,a_2^0)},$$
$$\beta_2=\left[\frac{\partial K^x}{\partial a_2}\right]_{(a_1^0,a_2^0)},\quad \beta_3=\left[\frac{\partial C^y}{\partial a_2}\right]_{(a_1^0,a_2^0)},\ldots \tag{22}$$

where (a_1^0,a_2^0) denotes the value of a_1, a_2, which yields the identity element of the group.

Now, we seek the absolute invariant of the independent variables. Owing to Eqs. (21), $\eta(x,y,t)$ is an absolute invariant if it satisfies the two first-order partial differential equations

$$(\alpha_1 x+\alpha_2)\frac{\partial\eta}{\partial x}+(\alpha_3 y)\frac{\partial\eta}{\partial y}+(\alpha_5 t+\alpha_6)\frac{\partial\eta}{\partial t}=0,$$
$$(\beta_1 x+\beta_2)\frac{\partial\eta}{\partial x}+(\beta_3 y)\frac{\partial\eta}{\partial y}+(\beta_5 t+\beta_6)\frac{\partial\eta}{\partial t}=0. \tag{23}$$

Since $K^y=0$, according to definition of $\alpha's$ and $\beta's$, we have

$$\alpha_4=\beta_4=0. \tag{24}$$

By successively eliminating $y\partial\eta/\partial y$ and $\partial\eta/\partial x$ in Eq. (23), we have

$$(\lambda_{13}x+\lambda_{23})\frac{\partial\eta}{\partial x}+(\lambda_{53}t+\lambda_{63})\frac{\partial\eta}{\partial t}=0,$$
$$(\lambda_{31}x+\lambda_{32})y\frac{\partial\eta}{\partial y}+(\lambda_{51}xt+\lambda_{61}x+\lambda_{52}t+\lambda_{62})\frac{\partial\eta}{\partial t}=0, \tag{25}$$

where

$$\lambda_{ij}=\alpha_i\beta_j-\alpha_j\beta_i,\quad i,j=1,2,\ldots,6. \tag{26}$$

According to the basic theorem of group theory, equation (23) has one and only one solution if the coefficient matrix has a rank two. The matrix has rank two whenever at least one of its two by two submatrices has a non-vanishing determinant; and this condition is met whenever at least one of the following is satisfied:

$$\lambda_{13}x+\lambda_{23}\neq 0,\ \text{ or }\ \lambda_{35}t+\lambda_{36}\neq 0,$$
$$\text{or }\ \lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26}\neq 0. \tag{27}$$

According to conditions (27), the following cases arise.

Case 1 The first case has been studied whenever one of the conditions of Eq. (27) vanished.

Sub-case 1.1 According to Eq. (27), we study the following sub-case:

$$\lambda_{31}x+\lambda_{32}=0,\quad \lambda_{35}t+\lambda_{36}\neq 0,$$
$$\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26}\neq 0. \tag{28}$$

Then from Eq. (25), $\partial\eta/\partial t=0$. This yields a solution $\eta=\eta(x,y)$. Then from Eq. (23),

$$\eta=y(Ax+B)^r,\quad r=\frac{-\alpha_3}{\alpha_1}=\frac{-\beta_3}{\beta_1},$$
$$A=\alpha_1=\beta_1,\quad B=\alpha_2=\beta_2. \tag{29}$$

Sub-case 1.2

$$\lambda_{31}x+\lambda_{32}\neq 0,\quad \lambda_{35}t+\lambda_{36}=0,$$
$$\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26}\neq 0. \tag{30}$$

From Eq. (25), $\partial\eta/\partial x = 0$. This yields a solution $\eta = \eta(y,t)$. Then from Eq. (23),

$$\eta = y(At+B)^r,\ \ r = \frac{-\alpha_3}{\alpha_5} = \frac{-\beta_3}{\beta_5},$$
$$A = \alpha_5 = \beta_5,\ \ B = \alpha_6 = \beta_6. \tag{31}$$

Sub-case 1.3

$$\lambda_{31}x+\lambda_{32} \neq 0,\ \ \lambda_{35}t+\lambda_{36} \neq 0,$$
$$\lambda_{15}xt+\lambda_{16}x+\lambda_{25}t+\lambda_{26} = 0, \tag{32}$$

then $\partial\eta/\partial y = 0$. This yields a solution $\eta = \eta(x,t)$, which is unacceptable with the boundary conditions.

Case 2 Here we suppose that some of the coefficients in Eq. (25) vanish.

Sub-case 2.1 We suppose that

$$\lambda_{31} = \lambda_{35} = \lambda_{15} = 0,\ \ \lambda_{32} \neq 0,\ \ \lambda_{36} \neq 0. \tag{33}$$

Then equation (25) has the solution

$$\eta = f(y,\zeta(x,t)),\ \ \zeta(x,t) = \lambda_{36}x - \lambda_{32}t = \text{const.}, \tag{34}$$

and f is an arbitrary function.

According to Eq. (25), the second equation has the form

$$y\frac{\partial f}{\partial y} - h(\zeta)\frac{\partial f}{\partial \zeta} = 0,\ \ h(\zeta) = \lambda_{16}x+\lambda_{25}t+\lambda_{26}. \tag{35}$$

Now, we are looking for a solution to Eq. (35) in the form

$$f = \phi(y\,H(\zeta)). \tag{36}$$

The function ϕ may be equal to one without loss of generality, after applying a mathematical technique

$$H(\zeta) = (\lambda_{16}x+\lambda_{25}t+\lambda_{26})^{-\lambda_{36}/\lambda_{16}},\ \ \lambda_{16} \neq 0. \tag{37}$$

From Eqs. (34), (36), and (37), the solution can be written in the form

$$\eta = y(Ax+Bt+C)^{-r},\ \ r = \frac{-\lambda_{36}}{\lambda_{61}},$$
$$A = \lambda_{16},\ \ B = \lambda_{25},\ \ C = \lambda_{26}. \tag{38}$$

Sub-case 2.2

$$\lambda_{35} = 0,\ \ \lambda_{31} \neq 0,\ \ \lambda_{36} \neq 0,\ \ \lambda_{23} \neq 0. \tag{39}$$

In the same manner, we obtain

$$\eta = y(x+A)^m\,\mathrm{e}^{-rt},\ \ A = \frac{\lambda_{23}}{\lambda_{13}},\ \ m = \lambda_{36},\ \ r = \lambda_{13}. \tag{40}$$

Sub-case 2.3

$$\lambda_{31} = 0,\ \ \lambda_{35} \neq 0,\ \ \lambda_{32} \neq 0. \tag{41}$$

In a similar manner, the formula of dimensionless coordinate η is given by

$$\eta = y(t+A)^m\,\mathrm{e}^{-rx},\ \ A = \frac{\lambda_{63}}{\lambda_{53}},\ \ m = -\lambda_{23},\ \ r = \lambda_{53}. \tag{42}$$

Sub-case 2.4

$$\lambda_{25} = 0,\ \ \lambda_{31} \neq 0,\ \ \lambda_{35} \neq 0, \lambda_{23} \neq 0,\ \ \lambda_{36} \neq 0. \tag{43}$$

Again, the independent absolute invariant η

$$\eta = y(t+A)(Bx+C)^r,\ \ A = \frac{\lambda_{36}}{\lambda_{35}},$$
$$B = \lambda_{13},\ \ C = \lambda_{23},\ \ r = -\frac{\lambda_{35}}{\lambda_{13}}. \tag{44}$$

Case 3

At last, we study the case which satisfies

$$\lambda_{31} = \lambda_{32} = \lambda_{36} = \lambda_{35} = \lambda_{15} = \lambda_{25} = 0. \tag{45}$$

This implies that

$$\frac{\partial\eta}{\partial t} = 0. \tag{46}$$

From Eq. (25),

$$\eta = f(y). \tag{47}$$

Without loss of generality, the independent absolute invariant $\eta(y)$ in Eq. (47) we may assume in the form

$$\eta = Ay. \tag{48}$$

For all previous cases, we can observe that, the independent absolute invariant

$$\eta = y\Pi(x,t), \tag{49}$$

where Π is a function deferent from case to another.

3.4. Absolute invariants of dependent variables

In the next step, we obtain the absolute invariants corresponding to the dependent variables. From Eq. (20), we mention that θ is itself an absolute invariant. Thus, $\theta(x,y,t) = \theta(\eta)$, equation (21) may be solved to obtain the other five absolute invariants. Frequently, the following forms corresponding to ψ, T_{w}, U, μ, and κ may be assumed as:

$$\psi(x,y,t) = \Gamma_1(x,t)f(\eta), \tag{50a}$$
$$T_1(x,y,t) = \Gamma_2(x,t)E(\eta), \tag{50b}$$
$$U(x,y,t) = \Gamma_3(x,t)H(\eta), \tag{50c}$$
$$\mu(x,y,t) = \Gamma_4(x,t)F_1(\eta), \tag{50d}$$
$$\kappa(x,y,t) = \Gamma_5(x,t)F_2(\eta). \tag{50e}$$

Since $U(x,t)$ and $T_1(x,t)$ are independent of y, whereas η depends on y, it follows that $E(\eta)$ and $H(\eta)$ must be equal to a constant, say T_0 and U_0, respectively. Thus, equation (50) becomes

$$\psi(x,y,t) = \Gamma_1(x,t)f(\eta), \tag{51a}$$
$$T_1(x,y,t) = \Gamma_2(x,t)T_0, \tag{51b}$$
$$U(x,y,t) = \Gamma_3(x,t)U_0, \tag{51c}$$
$$\mu(x,y,t) = \Gamma_4(x,t)F_1(\eta), \tag{51d}$$
$$\kappa(x,y,t) = \Gamma_5(x,t)F_2(\eta). \tag{51e}$$

The five functions Γ_1, Γ_2, ..., Γ_5 in Eqs. (51a)–(51e) respectively are those for which the governing equations (6) and (7) are reducible to ordinary differential equations.

3.5. Reduction to ordinary differential equations

As the general analysis proceeds, the established forms of the dependent and independent absolute invariant are used to obtain ordinary differential equations. Substituting Eqs. (51) into Eqs. (6)–(8), and invoking Eq. (49) yields

$$F_1'(f'')^n + nF_1 f'''(f'')^{n-1} - C_1(f'+\eta f'') - C_2(f'^2 - f'') - C_3(f')^2 - C_4 f' + C_5 + C_6 = 0, \tag{52}$$
$$\frac{1}{\mathrm{Pr}}(F_2'\theta' + F_2\theta'') - \eta C_7\theta' - C_8\theta + C_9 f\theta' - C_{10}\theta f' = 0, \tag{53}$$

with the boundary conditions

$$\eta = 0 : f' = 0,\ \ f = -\gamma,\ \ \theta = 1, \tag{54a}$$
$$\eta \to \infty : f' \to C_{13},\ \ \theta \to 0, \tag{54b}$$

where the primes refer to differentiation with respect to η and

$$C_1 = \frac{1}{(\Pi)^{2n+1}\Gamma_1^{n-1}\Gamma_4}\frac{\partial \Pi}{\partial t},\ \ C_2 = \frac{1}{(\Pi)^{2n-1}\Gamma_1^{n-1}\Gamma_4}\frac{\partial \Gamma_1}{\partial x},$$
$$C_3 = \frac{1}{(\Pi)^{2n}\Gamma_1^{n-2}\Gamma_4}\frac{\partial \Pi}{\partial x},\ \ C_4 = \frac{1}{(\Pi)^{2n}\Gamma_1^{n}\Gamma_4}\frac{\partial \Gamma_1}{\partial t},$$
$$C_5 = \frac{U_0}{(\Pi)^{2n+1}\Gamma_1^{n}\Gamma_4}\frac{\partial \Gamma_3}{\partial t},\ \ C_6 = \frac{U_0^2\Gamma_3}{(\Pi)^{2n+1}\Gamma_1^{n}\Gamma_4}\frac{\partial \Gamma_3}{\partial x},$$
$$C_7 = \frac{1}{(\Pi)^3\Gamma_5}\frac{\partial \Pi}{\partial t},\ \ C_8 = \frac{1}{(\Pi)^2\Gamma_2\Gamma_5}\frac{\partial \Gamma_2}{\partial t},$$
$$C_9 = \frac{1}{\Pi\Gamma_5}\frac{\partial \Gamma_1}{\partial x},\ \ C_{10} = \frac{\Gamma_1}{\Pi\Gamma_2\Gamma_5}\frac{\partial \Gamma_2}{\partial x},\ \ C_{11} = \frac{\partial \Gamma_1}{\partial x},$$
$$C_{12} = \frac{U_0\Gamma_3}{\Gamma_1\Pi},\ \ \gamma = \frac{-v_{\mathrm{w}}}{C_{11}}. \tag{55}$$

For equations (52)–(54) are reduced to ordinary differential equations, it is necessary that the coefficients C's must be constants or function of η only. It remains to utilize each of the η in turn with Eq. (55) to evaluate the appearing C's in the ordinary differential equations (52)–(54). Consequently, to evaluate the corresponding expressions of the functions Γ_1, Γ_2, Γ_3, Γ_4, and Γ_5.

3.5.1. Reduction to ordinary differential equations ($\eta = \Pi y, \Pi = A = 1$)

Here, we have some sub-cases of Γ_1 and Γ_2.

3.5.1.1 Reduction when $\Gamma_1 = \Gamma_1(x)$ and $\Gamma_2 = \Gamma_2(x)$

Then from Eq. (55),

$$C_1 = C_3 = C_4 = C_5 = C_7 = C_8 = 0,$$
$$C_2 = C_6 = C_9 = C_{11}. \tag{56}$$

By considering C_{12} as unity, we obtain

$$\Gamma_1 = C_2x + k_1,\ \ \Gamma_2 = A(C_2x+k_1)^m,\ \ \Gamma_3 = \frac{C_2x+k_1}{U_0}, \tag{57a}$$
$$\Gamma_4 = (C_2x+k_1)^{1-n},\ \ \Gamma_5 = 1,\ \ m = \frac{C_{10}}{C_2},\ \ C_2 \neq 0, \tag{57b}$$

where k_1 is an arbitrary constant. By substituting Eq. (57) into Eqs. (52)–(54), we have

$$nF_1 f'''(f'')^{n-1} + F_1'(f'')^n - C_2(f'^2 - ff'' - 1) = 0, \tag{58}$$
$$\frac{1}{\mathrm{Pr}}(F_2\theta'' + F_2'\theta') + C_2 f\theta' - C_{10}\theta f' = 0, \tag{59}$$

with the boundary conditions

$$\eta = 0 : f' = 0,\ \ f = -\gamma,\ \ \theta = 1, \tag{60a}$$
$$\eta \to \infty : f' \to 1,\ \ \theta \to 0. \tag{60b}$$

The forms of η, ψ, T_{w}, U, μ, and κ are given as follows:

$$\eta = y,\ \ \psi = (C_2x+k_1)f(\eta),$$
$$T_{\mathrm{w}} = T_\infty + T_0A(C_2x+k_1)^m, \tag{61a}$$
$$U = C_2x+k_1,\ \ \mu = (C_2x+k_1)^{1-n}F_1(\eta),$$
$$\kappa = F_2(\eta). \tag{61b}$$

By considering $C_{10} = 0$, then equations (58) and (59) become

$$nF_1 f'''(f'')^{n-1} + F_1'(f'')^n - C_2(f'^2 - ff'' - 1) = 0, \tag{62}$$
$$\frac{1}{\mathrm{Pr}}(F_2\theta'' + F_2'\theta') + C_2 f\theta' = 0, \tag{63}$$

with the boundary conditions given by Eq. (60) and $T_{\mathrm{w}} = \mathrm{const}$. If we choose F_1 and F_2 as $F_1 = (1+\alpha_1\theta)$ and $F_2 = (1+\alpha_2\theta)$, where α_1 is the viscosity parameter and α_2 is the thermal conductivity parameter. These parameters depend on the nature of the fluid, and when $n = 1$, $C_2 = 1$, $\alpha_1 = \alpha_2 = 0$, equations (58)–(60) are reduced to a special case of heat transfer for the Falkner–Skan flows.[20] Another special case, when $n = C_2 = 1$, $\alpha_1 = \alpha_2 = 0$, $\mathrm{Pr} = 1$, $m = C_{10} = \lambda$, where λ is the power index wall temperature parameter, equations (58)–(60) are reduced to a problem, which was solved by Yih,[21] and Tsia and Huang.[22]

3.5.1.2 Reduction when $\Gamma_1 = \Gamma_1(x)$ and $\Gamma_2 = \Gamma_2(t)$

Then from Eq. (55), we have

$$C_1 = C_3 = C_4 = C_5 = C_7 = C_{10} = 0,$$
$$C_2 = C_6 = C_9 = C_{11}. \tag{64}$$

By considering C_{12} as a unity, we obtain

$$\Gamma_1 = C_2 x + k_1,\ \ \Gamma_2 = A\,\mathrm{e}^{C_8 t},\ \ \Gamma_3 = \frac{C_2 x + k_1}{U_0},$$
$$\Gamma_4 = (C_2 x + k_1)^{1-n},\ \ \Gamma_5 = 1,\ \ C_2 \neq 0. \tag{65}$$

Differential equations (52) and (53) have the following form:

$$nF_1 f'''(f'')^{n-1} + F_1'(f'')^n - C_2(f'^2 - ff'' - 1) = 0, \tag{66}$$
$$\frac{1}{\mathrm{Pr}}(F_2\theta'' + F_2'\theta') + C_2 f\theta' - C_8\theta = 0, \tag{67}$$

with the boundary conditions (60). The invariant function has the forms of

$$\psi = (C_2 x + k_1) f(\eta),\ \ T_{\mathrm{w}} = T_\infty + T_0 A\,\mathrm{e}^{C_8 t}, \tag{68a}$$
$$U = C_2 x + k_1,\ \ \mu = (C_2 x + k_1)^{1-n} F_1(\eta),$$
$$\kappa = F_2(\eta). \tag{68b}$$

3.5.1.3. Reduction when $\Gamma_1 = \Gamma_1(t)$ and $\Gamma_2 = \Gamma_2(x)$

This sub-case occurs only when $v_{\mathrm{w}} = 0$ and $\Gamma_4 = \Gamma_5 = 1$. From Eq. (55),

$$C_1 = C_2 = C_3 = C_6 = C_7 = C_8 = C_9 = C_{10} = C_{11} = 0,$$
$$C_4 = C_5. \tag{69}$$

If C_{12} is a unity, we find after some mathematical steps that,

$$\Gamma_1 = ((C_4 t + k_1)(1-n))^{1-n},\ \ \Gamma_2 = \text{const.}, \tag{70a}$$
$$\Gamma_3 = \frac{((C_4 t + k_1)(1-n))^{1-n}}{U_0},\ \ \Gamma_4 = \Gamma_5 = 1,\ \ n \neq 1. \tag{70b}$$

Then, equations (52)–(54) will have the form of

$$nF_1 f'''(f'')^{n-1} + F_1'(f'')^n - C_4(f' - 1) = 0, \tag{71}$$
$$F_2\theta'' + F_2'\theta' = 0, \tag{72}$$

with the boundary conditions

$$\eta = 0: f' = 0,\ \ f = 0,\ \ \theta = 1, \tag{73a}$$
$$\eta \to \infty: f' \to 1,\ \ \theta \to 0. \tag{73b}$$

If $n = 1$, equations (52) and (53) will have the form

$$nF_1 f''' + F_1' f'' - C_4(f' - 1) = 0, \tag{74}$$
$$F_2\theta'' + F_2'\theta' = 0, \tag{75}$$

with the boundary conditions (73), and with

$$\Gamma_1 = k_1\,\mathrm{e}^{C_4 t},\ \ \Gamma_2 = \text{const.},\ \ \Gamma_3 = \frac{k_1}{U_0}\,\mathrm{e}^{C_4 t},\ \ \Gamma_4 = \Gamma_5 = 1. \tag{76}$$

3.5.1.4 Reduction when $\Gamma_1 = \Gamma_1(t)$ and $\Gamma_2 = \Gamma_2(t)$

This sub-case occurs only when $v_{\mathrm{w}} = 0$, $\Gamma_4 = \Gamma_5 = 1$, from Eq. (55)

$$C_1 = C_2 = C_3 = C_6 = C_7 = C_9 = C_{10} = C_{11} = 0,$$
$$C_4 = C_5,\ \ C_{12} = 1, \tag{77}$$

then

$$\psi = ((C_4 t + k_1)(1-n))^{1-n},\ \ T_{\mathrm{w}} = T_\infty + T_0 k_2\,\mathrm{e}^{C_8 t},$$
$$U = ((C_4 t + k_1)(1-n))^{1-n}, \tag{78a}$$
$$\mu = F_1(\eta),\ \ \kappa = F_2(\eta),\ \ n \neq 1. \tag{78b}$$

Then, equations (52) and (53) will have the form

$$nF_1 f'''(f'')^{n-1} + F_1'(f'')^n - C_4(f' - 1) = 0, \tag{79}$$
$$\frac{1}{\mathrm{Pr}}(F_2\theta'' + F_2'\theta') - C_8\theta = 0, \tag{80}$$

with the boundary conditions (73).

If $n = 1$, equations (52) and (53) will have the form

$$F_1 f''' + F_1' f'' - C_4(f' - 1) = 0, \tag{81}$$
$$\frac{1}{\mathrm{Pr}}(F_2\theta'' + F_2'\theta') - C_8\theta = 0, \tag{82}$$

with the boundary conditions (73), and with

$$\Gamma_1 = k_1\,\mathrm{e}^{C_4 t},\ \ \Gamma_2 = k_2\,\mathrm{e}^{C_8 t},$$
$$\Gamma_3 = \frac{k_1}{U_0}\,\mathrm{e}^{C_4 t},\ \ \Gamma_4 = \Gamma_5 = 1. \tag{83}$$

3.5.2. Reduction to ordinary differential equations ($\eta = \Pi y, \Pi = (At + B)^r$)

3.5.2.1 Reduction when $\Gamma_1 = \Gamma_1(t)$ and $\Gamma_2 = \Gamma_2(t)$

This sub-case occurs only when $v_{\mathrm{w}} = 0$, $r = -1$. From Eq. (55),

$$C_2 = C_3 = C_6 = C_9 = C_{10} = C_{11} = 0,$$
$$C_5 = C_4 + C_1,\ \ C_7 = -A. \tag{84}$$

If C_{12} is a unity, the forms of η, ψ, T_{w}, U, μ, and κ are given as follows:

$$\eta = \frac{y}{At + B},\ \ \psi = k_1(At + B)^{-C_4/C_1} f(\eta),$$
$$T_{\mathrm{w}} = T_\infty + T_0 k_2 (At + B)^{C_8/A}, \tag{85a}$$
$$U = k_1(At + B)^{-C_4/C_1 - 1},$$
$$\mu = \frac{-Ak_1^{1-n}}{C_1}(At + B)^{(2n-1)+(n-1)C_4/C_1} F_1(\eta),$$
$$\kappa = (At + B) F_1(\eta). \tag{85b}$$

Equations (52) and (53) become

$$nF_1 f'''(f'')^{n-1} + F_1'(f'')^n - C_1(f' - \eta f'' - 1)$$

$$+C_4(1-f')=0, \tag{86}$$

$$\frac{1}{\mathrm{Pr}}(F_2\theta''+F_2'\theta')+A\eta\theta'-C_8\theta=0, \tag{87}$$

with the boundary conditions (73).

3.5.2.2 Reduction when $\Gamma_1=\Gamma_1(x)$ and $\Gamma_2=\Gamma_2(x,t)$

It also may be remarked that from Eq. (55)

$$C_3=C_4=0,\ C_7=-A,\ C_2=C_6=\frac{-C_1C_{11}}{A},$$
$$C_5=C_1=0,\ r=-1. \tag{88}$$

If C_{12} is taken to be unity, the forms of η, ψ, T_{w}, U, μ, and κ are given as follows:

$$\eta=\frac{y}{At+B},\ \psi=(C_{11}x+k_1)f(\eta),$$
$$T_{\mathrm{w}}=T_\infty+T_0k_2(At+B)^{C_8/A}(C_{11}x+k_1)^{C_{10}/C_{11}}, \tag{89a}$$
$$U=\frac{C_{11}x+k_1}{At+B},\ \mu=\frac{-A(At+B)^{(2n-1)}}{C_1(C_{11}x+k_1)^{n-1}}F_1(\eta),$$
$$\kappa=(At+B)F_2(\eta). \tag{89b}$$

Then, equations (52) and (53) have the form

$$nF_1f'''(f'')^{n-1}+F_1'(f'')^n-C_1(f'+\eta f''-1)$$
$$+\frac{C_1C_{11}}{A}(f'^2-ff''-1)=0, \tag{90}$$
$$\frac{1}{\mathrm{Pr}}(F_2\theta''+F_2'\theta')+A\eta\theta'-C_8\theta$$
$$+C_{11}f\theta'-C_{10}f'\theta=0, \tag{91}$$

with the boundary conditions (60).

3.5.3. Reduction to ordinary differential equations ($\eta=\Pi y, \Pi=(Ax+B)^r$)

Here, we have the following conditions:

$$C_1=C_4=C_5=C_7=C_8=0,$$
$$C_9=C_{11}=A,\ C_3=C_6=C_2r, \tag{92}$$

and if $C_{12}=1$, the forms of η, ψ, T_{w}, U, μ, and κ are given as follows:

$$\eta=y(Ax+B)^r,\ \psi=(Ax+B)f(\eta),$$
$$T_{\mathrm{w}}=T_\infty+T_0k_2(Ax+B)^{C_{10}/A}, \tag{93a}$$
$$U=(Ax+B)^{r+1},\ \mu=\frac{A}{C_2}(Ax+B)^{r(1-2n)-n+1}F_1(\eta),$$
$$\kappa=(Ax+B)^{-r}F_2(\eta). \tag{93b}$$

The system of ordinary differential equations (52) and (53) reduces to

$$nF_1f'''(f'')^{n-1}+F_1'(f'')^n$$
$$-C_2(f'^2-ff''+rf'^2-r-1)=0, \tag{94}$$
$$\frac{1}{\mathrm{Pr}}(F_2\theta''+F_2'\theta')-Af\theta'-C_{10}f'\theta=0, \tag{95}$$

with the boundary conditions (60).

3.5.4. Reduction to ordinary differential equations ($\eta=\Pi y, \Pi=(Ax+Bt+c)^{-r}$)

For this case, it follows that

$$C_5=C_6=0, C_7=-B,\ C_3=-C_2=\frac{A}{B},$$
$$C_{10}=\frac{AC_8}{B},\ r=1, \tag{96}$$

which yields that

$$\eta=y(Ax+Bt+c)^{-1},\ \psi=\frac{C_{11}}{A}(Ax+Bt+c)f(\eta),$$
$$T_{\mathrm{w}}=T_\infty+T_0(Ax+Bt+c)^{C_8/B}, \tag{97a}$$
$$U=\frac{C_{11}}{A}=\mathrm{const.},\ \mu=\frac{-BA^{n-1}}{C_1C_{11}^{n-1}}(Ax+Bt+c)^nF_1(\eta),$$
$$\kappa=(Ax+Bt+c)F_2(\eta). \tag{97b}$$

Substituting the above equation into Eqs. (52) and (53), we obtain

$$nF_1f'''(f'')^{n-1}+F_1'(f'')^n+C_1\left(\eta f''+\frac{A}{B}ff''\right)=0, \tag{98}$$
$$\frac{1}{\mathrm{Pr}}(F_2\theta''+F_2'\theta')+C_{11}f\theta'-C_{10}f'\theta-C_8\theta=0, \tag{99}$$

with the boundary conditions (60).

3.5.5. Reduction to ordinary differential equations ($\eta=\Pi y, \Pi=(x+A)^m\mathrm{e}^{-rt}$)

The conditions (55) are satisfied only in the case of $r=0$, and this case is solved in Section (3.5.3).

3.5.6. Reduction to ordinary differential equations ($\eta=\Pi y, \Pi=(x+A)^m(t+B)^{-r}$)

The conditions (55) are satisfied only in the case of $r=-1$, $m=0$, and this case is solved in Section (3.5.2) with $A=1$.

The system of ordinary differential equations (6) and (7) with the boundary condition equations (8) has a comprehensive account of the form of similarity reduction. In the next section, we will solve two transformed systems of ordinary differential equations represent steady and unsteady cases.

4. Results and discussion

In order to validate our results, we have solved the set of non-linear ordinary equations (58) and (59) with boundary conditions (60) represents the similarity solution for the steady-state flow of a power-law fluid near a stagnation point with variable properties and variable wall temperature using the fourth-order Runge–Kutta integration with the shooting scheme. When $\gamma=-f_{\mathrm{w}}$, where the values of f_{w} are chosen to represent suction ($f_{\mathrm{w}}>0$), injection ($f_{\mathrm{w}}<0$) and ($f_{\mathrm{w}}=0$) for impermeable surface, and if we choose $C_{10}=\lambda$, $C_2=1$, and $F_1=F_2=n=1$, representative results for $f''(0)$ and $-\theta'(0)$

are compared with those of Yih,[21] and Tsai and Huang,[22] as shown in Tables 1 and 2, respectively. The results are found to be in a good agreement.

Table 1. Comparison of the values of $f''(0)$ with Pr = 1.

f_w	Yih[21] ($M=0$)	Tsai and Huang[22] ($M=\lambda=0$)	Present study ($\lambda=0$)
−1	0.75657	0.75650	0.756576
0	1.231	1.23257	1.23259
1	1.88931	1.88922	1.88932

In order to gain physical insight into the problem of steady state case, the wall shear stress $(f''(0))^n$ and the rate of heat transfer $\theta'(0)$ are discussed by assigning numerical values to the parameters encountered into the corresponding equations in Table 3, this by considering $F_1=(1+\alpha_1\theta)$ and $F_2=(1+\alpha_2\theta)$.

From Table 3 it can be seen that increasing Pr, λ, and f_w increase the wall shear stress as a result of increasing resistive forces, while by increasing n the wall shear stress increases for pseudo plastics fluids and decreases for dilatants. Moreover, increasing α_1 and α_2 decreases the wall shear stress. The effect of Pr is more pronounced on the magnitude of $\theta'(0)$. It is clear that increasing Pr increases the magnitude of $\theta'(0)$, this according to the well-known relation $\delta_\theta/\delta\approx \mathrm{Pr}^{-1/2}$, where δ_θ is the thickness of the thermal boundary layer and δ is the thickness of the velocity boundary layer. So, when the Prandtl number increases, the thickness of the thermal boundary layer becomes thinner and this causes an increase in the gradient of the temperature. Therefore, the absolute value of heat transfer coefficient $\theta'(0)$ increases as Pr increases. We also notice that increasing λ, f_w, and α_1 increase the magnitude of $\theta'(0)$, while as n and α_2 increase, the magnitude of $\theta'(0)$ decreases.

Table 2. Comparison of the values of $-\theta'(0)$ with $f_w=0$.

λ	Yih[21] ($M=0$) Pr = 1	Yih[21] ($M=0$) Pr = 10	Tsai and Huang[22] ($M=0$) Pr = 1	Tsai and Huang[22] ($M=0$) Pr = 10	Present study Pr = 1	Present study Pr = 10
0	0.570465	1.338796	0.570428	1.339367	0.570466	1.3388
1	0.811301	1.861577	0.811262	1.862357	0.811303	1.86159

Another case represents unsteady-state described by Eqs. (90) and (91) with boundary conditions (60) will be solved in this section, this by considering $f_w=-\gamma$, $C_{10}=\lambda$, $C_1=C_8=C_{11}=A=1$ and by considering $F_1=1+\alpha_1\theta$ and $F_2=1+\alpha_2\theta$. Table 4 illustrates the effects of all governing parameters on $(f''(0))^n$ and $-\theta(0)$ for unsteady case. It is clear that the wall shear stress increases $(f''(0))^n$ with increasing Pr, λ, f_w, n, α_1, while the thermal conductivity parameter α_2 has the opposite influence. The rate of heat transfer increases with increasing Pr, λ, f_w, and α_1, but it decreases with increasing n and α_2. The effect of f_w on $-\theta'(0)$ is that the rate of heat transfer decreases with blowing and increases with suction. This is because that the thermal boundary layer thickness increases (decreases) with injection (suction) which causes a decrease (increase) in the rate of heat transfer.

Figures 2–4 depict the respective changes in the dimensionless velocity as the variation of f_w, n, and α_1 respectively. Figure 2 shows the effects of suction $f_w>0$ and injection $f_w<0$ on the dimensionless velocity f', the effect of suction is to decrease the velocity whereas the effect of injection is to increase this. The physical explanation for such a behavior is that suction stabilizes the boundary layer growth, while injection increases the velocity in the boundary layer region, indicating that injection helps the flow penetrate more into the fluid. The variations of f' with the generalized power-law viscosity index n is illustrated in Fig. 3, the dimensionless velocity f' at the wall is decreasing with an increase in

Table 3. The effect of dynamic parameters on wall shear stress, and the rate of heat transfer for steady state case.

Pr	λ	f_w	n	α_1	α_2	$(f''(0))^n$	$-\theta'(0)$
1	0.7	2	1.2	–0.3	0.4	3.70428	1.72981
3	0.7	2	1.2	–0.3	0.4	3.76003	4.50257
7	0.7	2	1.2	–0.3	0.4	3.79597	10.1253
10	0.7	2	1.2	–0.3	0.4	3.80626	14.3795
10	0	0.2	0.2	–0.04	1	1.40245	1.82928
10	0.5	0.2	0.2	–0.04	1	1.40367	2.08968
10	1	0.2	0.2	–0.04	1	1.40463	2.30396
10	1.5	0.2	0.2	–0.04	1	1.40542	2.48756
10	0.1	–0.8	0.8	–0.5	0.1	1.05662	0.0200636
10	0.1	–0.5	0.8	–0.5	0.1	1.33272	0.110626
10	0.1	0	0.8	–0.5	0.1	2.06445	1.53026
10	0.1	2	0.8	–0.5	0.1	5.21889	18.291101
10	1	1	0.2	–0.2	0.01	2.25467	10.927
10	1	1	0.5	–0.2	0.01	2.32648	10.3914
10	1	1	0.8	–0.2	0.01	2.33778	10.2519
10	1	1	1	–0.2	0.01	2.33581	10.2102
10	1	1	1.2	–0.2	0.01	2.33225	10.1842
10	1	1	1.6	–0.2	0.01	2.32523	10.154
10	0.5	–0.2	0.5	0	0.2	1.21279	0.802743
10	0.5	–0.2	0.5	–0.1	0.2	1.28196	0.839701
10	0.5	–0.2	0.5	–0.6	0.2	1.93515	1.11662
10	0.5	–0.2	0.5	–0.9	0.2	3.61642	1.49737
10	0.3	0.4	0.6	–0.1	0	1.65519	4.65775
10	0.3	0.4	0.6	–0.1	0.1	1.6541	4.27083
10	0.3	0.4	0.6	–0.1	1	1.64604	2.53342
10	0.3	0.4	0.6	–0.1	2	1.63969	1.82456

power-law viscosity index n. This behavior is convenient with the fact that the pseudo-plastic fluids are more amenable to flow than the dilatant fluids. The situation is reversed far away from the wall. The graph for the velocity f' is plotted for different values of the viscosity parameter α_1 in Fig. 4, it depict that the velocity f' increases with increasing the magnitude values of α_1, and the fluid viscosity decreases the increment of the boundary layer thickness.

Table 4. The effect of dynamic parameters on wall shear stress, and the rate of heat transfer for unsteady case.

Pr	λ	f_w	n	α_1	α_2	$(f''(0))^n$	$-\theta'(0)$
1	0.5	–1	0.8	–0.1	0.4	0.591867	0.741625
3	0.5	–1	0.8	–0.1	0.4	0.600488	0.9261179
7	0.5	–1	0.8	–0.1	0.4	0.603335	0.991972
10	0.5	–1	0.8	–0.1	0.4	0.604095	0.999201
10	0	0.5	1.2	–0.5	0.02	2.72766	7.39924
10	0.2	0.5	1.2	–0.5	0.02	2.72811	7.42609
10	0.5	0.5	1.2	–0.5	0.02	2.7288	7.46563
10	1	0.5	1.2	–0.5	0.02	2.72987	7.5297
10	0.3	–1	0.5	–0.9	1	1.12191	0.991284
10	0.3	–0.5	0.5	–0.9	1	3.47422	2.16434
10	0.3	0	0.5	–0.9	1	5.25803	2.96848
10	0.3	0.5	0.5	–0.9	1	8.99397	4.63658
10	0.3	1	0.5	–0.9	1	13.2508	6.6019
10	1	1	0.5	–0.4	0.01	2.91859	11.8707
10	1	1	0.8	–0.4	0.01	2.94336	11.7437
10	1	1	1	–0.4	0.01	2.95158	11.7089
10	1	1	1.2	–0.4	0.01	2.95852	11.6881
10	1	1	1.6	–0.4	0.01	2.97055	11.6645
10	0.3	–0.5	0.2	0	0.1	0.972643	1.90363
10	0.3	–0.5	0.2	–0.1	0.1	1.02342	1.92311
10	0.3	–0.5	0.2	–0.5	0.1	1.24487	2.0388
10	0.3	–0.5	0.2	–0.8	0.1	1.39919	2.1628
10	0.1	0.8	1.2	–0.2	0	2.03922	9.9721
10	0.1	0.8	1.2	–0.2	0.1	2.03844	9.146113
10	0.1	0.8	1.2	–0.2	1	2.03187	5.40975
10	0.1	0.8	1.2	–0.2	1.5	2.02861	4.4852

Figures 5–8 illustrate the influence of the physical parameters f_w, α_1, α_2, and Pr on the dimensionless temperature profiles θ respectively. From Fig. 5 it is noticed that the suction parameter has the effect of reduction the dimensionless temperature in the boundary layer region while the absolute value of the blowing parameter enhancing it. This means that the thermal boundary layer thickness increases with injection and decreases with suction. This emphasizes the usual fact that suction stabilizes the boundary layer growth. The blowing parameter has exactly the opposite effect. Figures 6 and 7 shows the effect of the viscosity parameter α_1 and the thermal conductivity parameter α_2 on the temperature. From these figures we see that the temperature decreases with increasing magnitude of α_1 and increases with increasing α_2. Figure 8 depicts the effect of the Prandtl number Pr on the temperature profiles. As mentioned before, when the Prandtl number increases the thickness of the thermal boundary layer increases. Hence, the temperature θ increases with the decrease in Pr.

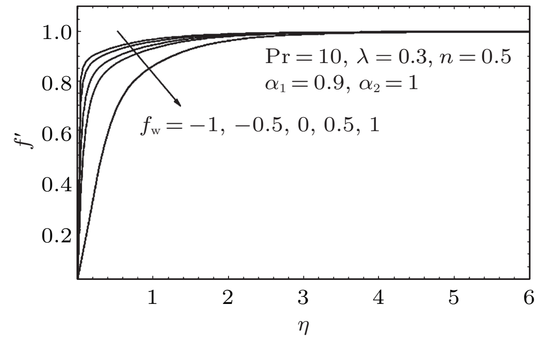

Fig. 2. Velocity distribution for various values of f_w.

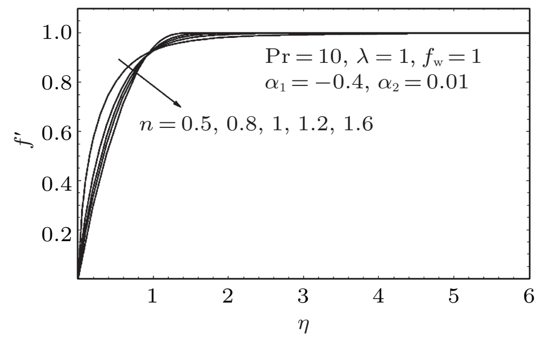

Fig. 3. Velocity distribution for various values of n.

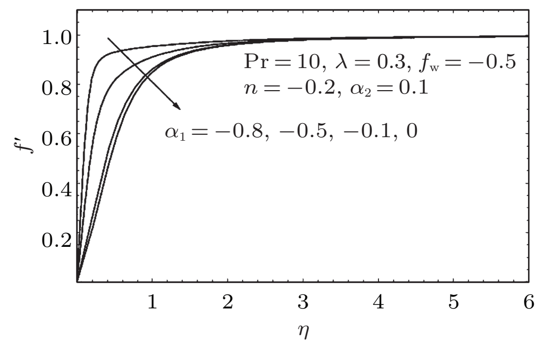

Fig. 4. Velocity distribution for various values of α_1.

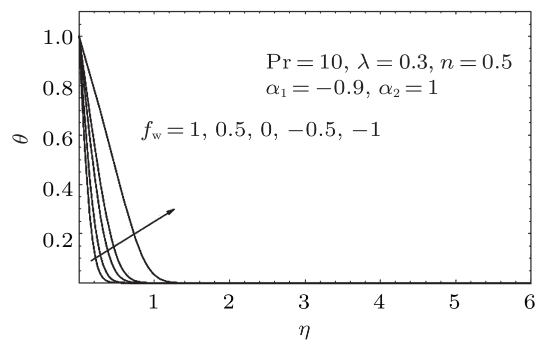

Fig. 5. Temperature distribution for various values of f_w.

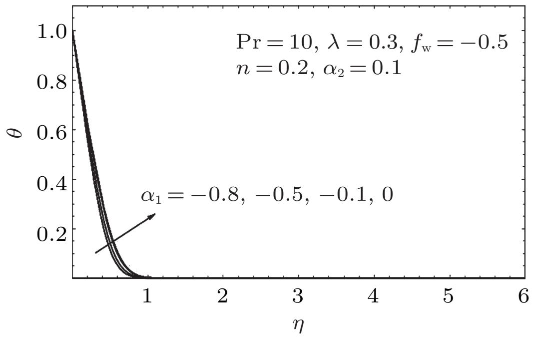

Fig. 6. Temperature distribution for various values of α_1.

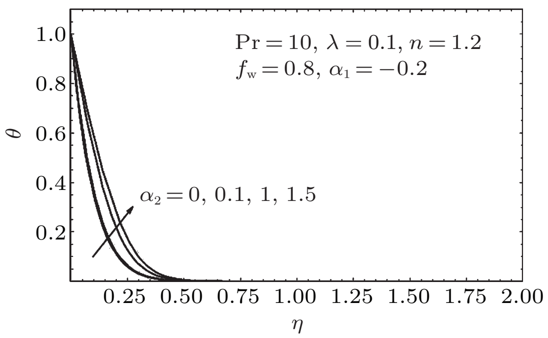

Fig. 7. Temperature distribution for various values of α_2.

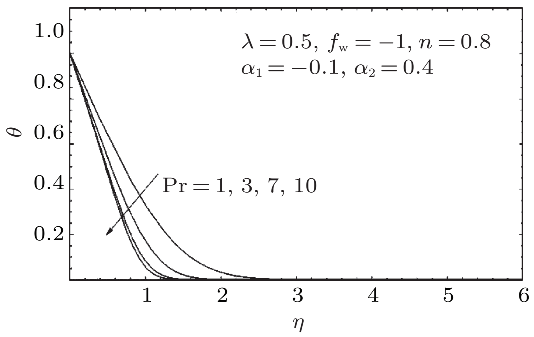

Fig. 8. Temperature distribution for various values of Pr.

5. Conclusions

A solution methodology based on the group theoretic method has been applied to solve the problem of unsteady power-law fluid near an unsteady stagnation point with variable viscosity and thermal conductivity, suction/blowing and with variable surface temperature. We have derived eight similarity solutions that reduce the nonlinear partial differential equations to ordinary differential equations. Two cases of differential equations with the associated boundary conditions represent steady and unsteady cases have been solved numerically using the shooting method, in which we used the quality controlled fourth-order Runge–Kutta technique.

For unsteady state case, it was found that the velocity f' decreases with suction, while it increases with injection and magnitude values of viscosity parameter. Also the temperature θ decreases as Prandtl number, suction parameter, and the magnitude of viscosity parameter increase. It increase as the thermal conductivity parameter and the absolute value of the blowing parameter increase. Moreover, it is found that the wall shear stress increases with Prandtl number or the power index wall temperature parameter or viscosity parameter while it decreases with the thermal conductivity parameter. However, the rate of heat transfer increases with suction and decreases with injection.

In the future, there is a possibility to develop the present work by using the Homotopy perturbation method coupling with the group theory. We think that it will be worth applying these methods together.[23–27]

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